Appendix B: Filtering and Subsidy Repeal

Utility is $\theta s_i + x_i$ where θ denotes taste for housing quality, s_i indexes housing segment i's quality and x_i is the numeraire. Taste θ is distributed according to cdf F, on support $[a, b]$ with $a < b$. Rent in segment i is $q_i - \sigma_i$, where σ_i is the subsidy that may apply to i. Hence utility becomes $\theta_{s_i} + \omega - (q_i - \sigma_i)$ when residing in segment i. There are n city residents altogether (where now n is set to 1, for simplicity). Each household picks the quality that suits it best. We identify the threshold tastes $\underline{\theta}$ and $\bar{\theta}$ – owners of which are indifferent between segments 1 and 2, and between 2 and 3, respectively – as

$$\underline{\theta}(q_1, q_2 - \sigma_2) = (q_2 - \sigma_2 - q_1)/(s_2 - s_1) \quad [B1]$$
$$\bar{\theta}(q_2 - \sigma_2, q_3 - \sigma_3) = \left(\bar{q}_3 - q_2 - (\sigma_3 - \sigma_2)\right)/(s_3 - s_2). \quad [B2]$$

We let $\underline{\theta}_1$ denote the derivative of $\underline{\theta}$ with respect to q_1, $\underline{\theta}_2$ the derivative of $\underline{\theta}$ with respect to $q_2 - \sigma_2$, and so on. We note that $\overline{\theta}_2 = -\overline{\theta}_3$.

In an interior equilibrium, households with taste in $\left[a, \underline{\theta}\right]$ sort into rental housing, those (with tastes) in $\left(\underline{\theta}, \overline{\theta}\right]$ sort into existing homes, and those in $\left(\overline{\theta}, b\right]$ opt for a new home. Individual choices translate into aggregate housing demands, equal to $n_1 = F\left(\underline{\theta}\right), n_2 = \left(F\left(\overline{\theta}\right) - F\left(\underline{\theta}\right)\right)$ and $n_3 = \left(1 - F\left(\overline{\theta}\right)\right)$. Let n_{ij} denote the derivative of aggregate housing demand for housing quality i with respect to price j. The following properties apply

$$n_{11} = f\left(\underline{\theta}\right)\underline{\theta}_1 < 0 \quad \text{and} \quad n_{12} = f\left(\underline{\theta}\right)\underline{\theta}_2 > 0 \quad [B3]$$

$$n_{21} = -f\left(\underline{\theta}\right)\underline{\theta}_1 > 0 \quad \text{and} \quad n_{22} = \left(f\left(\overline{\theta}\right)\overline{\theta}_2 - f\left(\underline{\theta}\right)\underline{\theta}_2\right) < 0 \quad [B4]$$

$$\text{and} \quad n_{23} = -f\left(\overline{\theta}\right)\overline{\theta}_3 > 0 \quad [B5]$$

to the extent that $n_{11} + n_{21} = 0 > n_{12} + n_{22}$.

New homes are supplied outside the city center, in the periphery, only.[1] Space constraints have much less of a role in the periphery, and so we take the liberty to assume that new homes are supplied perfectly elastically at constant marginal cost $\bar{q}_3$. In this segment suppliers satisfy any demand at price $\bar{q}_3$. The cum-subsidy (i.e., consumer) price becomes $\bar{q}_3 - \sigma_3$. We set the equilibrium conditions for the inter-connected segments of apartments and existing homes as follows.

$$n_1(q_1, q_2 - \sigma_2) = s_1(q_1)$$
$$n_2\left(q_1, q_2 - \sigma_2, \overline{q}_3 - \sigma_3\right) = s_2(q_2) \quad [B6]$$

where s_i is supply in segment i (never at risk of confusion with quality s_i as we suppress the quality index in what follows). For consistency, increases in s_2 (following increases in

[1] Glaeser (2011) suggests as much, emphasizing the coincidence of owner-occupied housing with peripheral location for the US. Ahlfeldt and Maennig (2015) observe strong positive correlation between a ring's share of owner-occupiers and its distance to the city center for Berlin.

q_2) come about as existing vacant housing is supplied more; while increases in s_3 (following increases in q_3) we interpret as new construction. Let s_{ii} denote supply i's (strictly positive) derivative with respect to its own price below.

We translate Germany's full EZ-subsidy removal into policy changes $d\sigma_2 = -\sigma_2 < 0$ and $d\sigma_3 = -\sigma_3 < 0$ where $\sigma_2 < \sigma_3$.[2] We are interested in these policy changes' effects on qualities' prices and quantities, and on the distribution of city population across all three qualities. Removing the subsidy for new homes reduces equilibrium demand in that segment. But changes in the neighboring two segments are less obvious. To sort out the filtering flows involved, totally differentiate the equilibrium, keep in mind $d\bar{q}_3 = 0$, and rearrange to give

$$\begin{pmatrix} n_{11} - s_{11} & n_{12} \\ n_{21} & n_{22} - s_{22} \end{pmatrix} \begin{pmatrix} dq_1 \\ dq_2 \end{pmatrix} = \begin{pmatrix} n_{12}\, d\sigma_2 \\ n_{22}\, d\sigma_2 + n_{23}\, d\sigma_3 \end{pmatrix} \tag{B7}$$

or $A\, dq = db$ for short. Immediately we see that $|A| = (n_{11} - s_{11})(n_{22} - s_{22}) - n_{21}n_{12}$ is ambiguous in sign, and so with no further assumption nothing can be said about the sign of dq_1.

And then, the coefficient matrix A has three features we have not exploited yet. The first of these is its dominant diagonal, easily verified by summing all elements of a column and exploiting Equation [B3] or [B4]. Already we conclude that A's inverse has negative entries only (Sweeney (1974)). Two more of A's properties obtain once we rewrite matrix inverse A^{-1} as $G = (g_{ij})_{i,j=1,2}$. For G it must be true that $g_{11} < g_{12}$ as well as $g_{22} < g_{21}$. To these inequalities we refer to as "Sweeney's first and second property" below.[3]

Write the solution to the differentiated system of equilibrium equations as $\mathrm{dq} = \mathrm{A}^{-1}\mathrm{db}$. The price change in segment 1, dq_1, can then be rewritten as

$$\begin{aligned} dq_1 &= g_{11}\, n_{12}\, d\sigma_2 + g_{12}\, n_{22}\, d\sigma_2 + g_{12}\, n_{23}\, d\sigma_3 \\ &= \underbrace{f(\underline{\theta})\underline{\theta}_2 d\sigma_2}_{-} \left(\underbrace{g_{11} - g_{12}}_{-} \right) + \underbrace{g_{12}\, f(\overline{\theta})\, \overline{\theta}_2}_{+} \left(\underbrace{d\sigma_2 - d\sigma_3}_{+} \right) > 0 \end{aligned} \tag{B8}$$

where the first and last term on the first line of [B8] are positive, while the second term on the line is negative. And yet we are able, after signing all individual terms on the second line of [B8], to also sign dq_1 as positive, nonetheless.

Replacing n_{12}, n_{22} and n_{23} on the first line of [B8] by making use of [B3] through [B5], exploiting $\bar{\theta}_2 = -\bar{\theta}_3$, and rearranging translates into the second line of [B8]. Given Sweeney's first property, i.e., $g_{11} < g_{12}$, the first term on the right-hand side of the second line of [B8] must be positive. Moreover, given the structure of subsidy phase-out, i.e., $d\sigma_3 < d\sigma_2$, the second term on the right-hand side of [B8] is positive also. Thus $0 < dq_1$.

[2] These changes are not "small", and so our emphasis below is on direction, and not so much size, of the endogenous changes implied.

[3] These inequalities are implied by Sweeney's (1974) general "commodity hierarchy"-type preferences (of which ours are a special case). They are easily shown when recalling that $A^{-1}A = I$ and exploiting the two component equations corresponding to the two zero entries of the identity matrix. For example, $g_{21}(n_{11} - s_{11}) + g_{22}n_{21} = 0$ and hence $g_{21}/g_{22} < 1$.

Lifting both of EZ's component subsidies does raise the price of rental housing. (Note how this result hinges on being able to sign $(d\sigma_2 - d\sigma_3)$.) Now, because $s_{11} > 0$, apartment supply must have risen, too, as must have equilibrium rental housing demand. Hence $d\underline{\theta} > 0$. Yet $d\underline{\theta} > 0$ in turn implies that $dq_1 < d(q_2 - \sigma_2)$. Recalling $-d\sigma_2 < -d\sigma_3$, we conclude that all three qualities' (consumer) prices have gone up, and that

$$0 < dq_1 < d(q_2 - \sigma_2) < d(\overline{q}_3 - \sigma_3). \qquad [B9]$$

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