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Appendix A: Derivation of the Second-Order WTP Approximation

Let $V(q, p, m)$ be indirect utility in terms of environmental quality, prices of private goods and services and income respectively. We can define the marginal benefit or shadow value for changes in environmental quality in (1).

$$(1)\ \ b = \frac{V_q}{V_m}$$

Derivatives of b with respect to quality, income and prices are as follows

$$(2a)\ \ b_q = \frac{V_{qq}}{V_m} - b\,\frac{V_{qm}}{V_m}$$

$$(2b)\ \ b_m = \frac{V_{mq}}{V_m} - b\,\frac{V_{mm}}{V_m}$$

$$(2c)\ \ b_p = \frac{V_{qp}}{V_m} - b\,\frac{V_{mp}}{V_m}$$

Equation (3) defines WTP (labeled as W) for a change in q from q_0 to $q(t)$.

$$(3)\ \ V(q(t), p, m - W(t)) = V(q_0, p, m)$$

Develop the second order approximation for $W(t)$ evaluated at $t = 0$. Let $q(t) = q_0 + \alpha t$, with $\alpha = q_1 - q_0$.

$$(4)\ \ V\big(q_0 + \alpha t, p, m - W(t)\big) = V(q_0, p, m)$$

Differentiate (4) with respect to t:

$$(5)\ \ V_q\alpha - V_m W' = 0$$

Differentiate again with respect to t:

$$(6)\ \ V_{qq}\alpha^2 - V_{qm}\alpha W' - V_{mq}W'\alpha + V_{mm}(W')^2 - V_m W'' = 0$$

The second order expansions for $W(t)$ evaluated at $t = 0$ is

$$(7)\ \ W(t) \approx W(0) + W'(0)t + \frac{1}{2}W''(0)t^2$$

Solving for $W'(0)$ and $W''(0)$ from equations (5) and (6), we have:

$$\text{(8a)}\quad W'(0) = \frac{V_q}{V_m}\alpha = \frac{V_q}{V_m}(q_1 - q_0) = b(q_1 - q_0)$$

$$\text{(8b)}\quad W''(0) = \frac{V_{qq}}{V_m}\alpha^2 - 2\frac{V_{mq}}{V_m}\frac{V_q}{V_m}\alpha^2 + \frac{V_{mm}}{V_m}(\frac{V_q}{V_m})^2\alpha^2$$

We can re-write $W''(0)$ in terms of derivatives of b.

$$\text{(9)}\quad W''(0) = (b_q - bb_m)\alpha^2$$

Recognizing that $W(0) = 0$, our second order expansion of $W(t)$ evaluated at $t = 0$, with $t = 1$ is:

$$\text{(10)}\quad W(1) \approx b(q_1 - q_0) + \frac{1}{2}(b_q - bb_m)(q_1 - q_0)^2$$

Now to expansion in both p and q, with $p(t) = p_0 + \beta t$ and $\beta = p_1 - p_0$.

To simplify the notation, we assume there is only one private good (x).

$$\text{(11)}\quad x = -\frac{V_p}{V_m}$$

Derivatives of x with respect to price and income are as follows:

$$\text{(12a)}\quad x_p = -\frac{V_{pp}}{V_m} - x\frac{V_{mp}}{V_m}$$

$$\text{(12b)}\quad x_m = -\frac{V_{pm}}{V_m} - x\frac{V_{mm}}{V_m}$$

Equation (13) defines WTP (labeled as w) for both quality and price changes.

$$\text{(13)}\quad V\big(q_0 + \alpha t, p_0 + \beta t, m - w(t)\big) = V(q_0, p_0, m)$$

Differentiate (13) with respect to t:

$$\text{(14)}\quad V_q\alpha + V_p\beta - V_m w' = 0$$

Differentiate again with respect to t:

$$\text{(15)}\quad \begin{bmatrix} V_{qq}\alpha^2 + V_{qp}\alpha\beta - V_{qm}\alpha w' + V_{pp}\beta^2 + V_{pq}\alpha\beta - \\ V_{pm}\beta w' - V_{qm}\alpha w' - V_{pm}\beta w' + V_{mm}(w')^2 - V_m w'' \end{bmatrix} = 0$$

Solving for $w'(0)$ and $w''(0)$ from equations (14) and (15), we have:

$$\text{(16a)}\quad w'(0) = \frac{V_q}{V_m}\alpha + \frac{V_p}{V_m}\beta = b\alpha - x\beta$$

$$\text{(16b)}\quad w''(0) = \frac{V_{qq}}{V_m}\alpha^2 + 2\frac{V_{qp}}{V_m}\alpha\beta - 2\frac{V_{qm}}{V_m}w'\alpha - 2\frac{V_{pm}}{V_m}w'\beta + \frac{V_{pp}}{V_m}\beta^2 + \frac{V_{mm}}{V_m}(w')^2$$

Substituting $w'(0)$ in equation (16b), we have

$$\text{(17)}\quad w''(0) = \frac{V_{qq}}{V_m}\alpha^2 + 2\frac{V_{qp}}{V_m}\alpha\beta - 2\frac{V_{qm}}{V_m}(b\alpha^2 - x\alpha\beta) - 2\frac{V_{pm}}{V_m}(b\alpha\beta - x\beta^2) + \frac{V_{pp}}{V_m}\beta^2 + \frac{V_{mm}}{V_m}(b^2\alpha^2 - 2x\beta b\alpha + x^2\beta^2)$$

Or

$$\text{(17a)}\quad w''(0) = (b_q - bb_m)\alpha^2 - (x_p + xx_m)\beta^2 + 2(b_p + xb_m)\alpha\beta$$

By evaluating the partial derivatives at $t = 0$, we have a second order approximation for $w(t)$ at $t = 1$:

$$\text{(18)}\quad w(1) \approx b(q_1 - q_0) - x(p_1 - p_0) + \frac{1}{2}((b_q - bb_m)(q_1 - q_0)^2 - (x_p + xx_m)(p_1 - p_0)^2 + 2(b_p + xb_m)(q_1 - q_0)(p_1 - p_0))$$