Appendix

A1. Derivation of equation [6]

We use two equations to solve for changes in the virtual magnitudes: the definition of virtual full income

$$\mu(p_1, p_2, w, \tilde{X}_3) = w + \pi_3(p_1, p_2, w, \tilde{X}_3)\tilde{X}_3$$

and the constraint on the level of the public good

$$X_3(p_1, p_2, \pi_3(\cdot), \mu(\cdot)) = \tilde{X}_3.$$

Differentiating both of these with respect to $\tilde{X}_3$ yields

$$\frac{\partial \mu}{\partial \tilde{X}_3} = \frac{\partial \pi_3}{\partial \tilde{X}_3}\tilde{X}_3 + \pi_3,$$

and

$$X_{33}\frac{\partial \pi_3}{\partial \tilde{X}_3} + X_{3\mu}\frac{\partial \mu}{\partial \tilde{X}_3} = 1.$$

We can obtain the following matrix representation for this system of equations:

$$\begin{bmatrix} X_{33} & X_{3\mu} \\ \tilde{X}_3 & -1 \end{bmatrix} \begin{bmatrix} \frac{\partial \pi_3}{\partial \tilde{X}_3} \\ \frac{\partial \mu}{\partial \tilde{X}_3} \end{bmatrix} = \begin{bmatrix} 1 \\ -\pi_3 \end{bmatrix}.$$

Using Cramer's rule, we can solve for

$$\frac{\partial \pi_3}{\partial \tilde{X}_3} = \frac{\det\begin{bmatrix} 1 & X_{3\mu} \\ -\pi_3 & -1 \end{bmatrix}}{\det\begin{bmatrix} X_{33} & X_{3\mu} \\ \tilde{X}_3 & -1 \end{bmatrix}} = \frac{-1 + \pi_3 X_{3\mu}}{-X_{33} - X_3 X_{3\mu}}$$

$$\frac{\partial \mu}{\partial \tilde{X}_3} = \frac{\det\begin{bmatrix} X_{33} & 1 \\ \tilde{X}_3 & -\pi_3 \end{bmatrix}}{\det\begin{bmatrix} X_{33} & X_{3\mu} \\ \tilde{X}_3 & -1 \end{bmatrix}} = \frac{-X_{33}\pi_3 - \tilde{X}_3}{-X_{33} - X_3 X_{3\mu}} = \frac{X_{33}\pi_3 + \tilde{X}_3}{X_{33} + X_3 X_{3\mu}}.$$

We can further simplify these expressions by substituting in the Slutsky decomposition, $X_{33} = C_{33} - X_3 X_{3\mu}$, to obtain

$$\begin{aligned}
\frac{\partial \pi_3}{\partial \tilde{X}_3} &= \frac{-1 + \pi_3 X_{3\mu}}{-X_{33} - X_3 X_{3\mu}} \\
&= \frac{-1 + \pi_3 X_{3\mu}}{-C_{33}} \\
&= \frac{1 - \pi_3 X_{3\mu}}{C_{33}} \\
\frac{\partial \mu}{\partial \tilde{X}_3} &= \frac{X_{33}\pi_3 + \tilde{X}_3}{X_{33} + X_3 X_{3\mu}} \\
&= \frac{\left(C_{33} - X_3 X_{3\mu}\right)\pi_3 + \tilde{X}_3}{C_{33}} \\
&= \frac{X_3 X_{3\mu}\pi_3 - \tilde{X}_3}{-C_{33}} + \pi_3 \\
&= \left[\frac{1 - \pi_3 X_{3\mu}}{C_{33}}\right]\tilde{X}_3 + \pi_3.
\end{aligned}$$

Note that the last step uses the constraint $X_3 = \tilde{X}_3$. Substituting these expressions into [5] and rearranging immediately yields [6].

To show this, we have from [5] that

$$\begin{aligned}
\frac{\partial \hat{x}_j\left(p_1, p_2, w, \tilde{X}_3\right)}{\partial \tilde{X}_3} &= \left[C_{j3} - X_3 x_{j\mu}\right]\frac{\partial \pi_3}{\partial \tilde{X}_3} + x_{j\mu}\frac{\partial \mu}{\partial \tilde{X}_3} \\
&= \left[C_{j3} - X_3 x_{j\mu}\right]\left(\frac{1 - \pi_3 X_{3\mu}}{C_{33}}\right) + x_{j\mu}\left(\left[\frac{1 - \pi_3 X_{3\mu}}{C_{33}}\right]\tilde{X}_3 + \pi_3\right) \\
&= C_{j3}\left(\frac{1 - \pi_3 X_{3\mu}}{C_{33}}\right) - X_3 x_{j\mu}\left(\frac{1 - \pi_3 X_{3\mu}}{C_{33}}\right) + x_{j\mu}\left[\frac{1 - \pi_3 X_{3\mu}}{C_{33}}\right]\tilde{X}_3 + \pi_3 x_{j\mu}.
\end{aligned}$$

Using the constraint $X_3 = \tilde{X}_3$, we can cancel the middle terms to obtain [6]

$$\frac{\partial \hat{x}_j\left(p_1, p_2, w, \tilde{X}_3\right)}{\partial \tilde{X}_3} = \frac{C_{j3}\left(1 - \pi_3 X_{3\mu}\right)}{C_{33}} + \pi_3 x_{j\mu}.$$

A2. Derivation of conditions [7] and [8]

Allowing heterogeneous preferences, endowments, and arbitrary welfare weights a_i for all i, the set of Pareto optimal allocations must solve

$$\max_{\{x_1\}^i,\{x_2\}^i} \sum_{i=1}^{n} a_i\, U^i\left(x_1^i, x_2^i, Y\right) \text{ s.t. } Y = \sum_{i=1}^{n}\left(w^i - p_1 x_1^i - p_2 x_2^i\right).$$

The first-order conditions are

$$\begin{aligned} \sum_{i=1}^{n} a_i\, p_1 U_3^i &= a_i U_1^i \text{ for all } i \\ \sum_{i=1}^{n} a_i\, p_2 U_3^i &= a_i U_2^i \text{ for all } i. \end{aligned}$$

The first implies that $a_i U_1^i$ is equal for all i, and the second implies that $a_i U_2^i$ is equal for all i. With a bit of rearranging, these first-order conditions immediately imply the conditions in [7]. Assuming further that all individuals have identical preferences and that $a_i = a$ for all i, the first-order conditions above further imply the conditions in [8], which define the symmetric, Pareto optimal allocation.

A3. Derivation of equation [13]

The solution to [11] will satisfy the unrestricted solution based on virtual magnitudes such that

$$\hat{x}_2(t_2, \tilde{X}_3) = x_2(\pi_2(\cdot), \pi_3(\cdot), \mu(\cdot)), \qquad \text{[A1]}$$

where the virtual magnitudes are themselves functions of the exogenous parameters $(p_1, p_2, w, \tilde{X}_3, t_2)$. π_2 is defined analogously to π_3 in the main text; it is the virtual price of x_2, and it newly appears because the dedicated tax leads to joint production of x_2 and x_3. It will, however, cancel out in the steps below. Differentiating equation [A1], the comparative static of interest can be expressed generally as

$$\frac{\partial \hat{x}_2}{\partial t_2} = x_{22}\frac{\partial \pi_2}{\partial t_2} + x_{23}\frac{\partial \pi_3}{\partial t_2} + x_{2\mu}\frac{\partial \mu}{\partial t_2}. \qquad \text{[A2]}$$

Following the method outlined in Cornes and Sandler (1996, p262,295), the three equations that the virtual magnitudes must satisfy are based on the tax-inclusive price of x_2 equaling the value of what it buys in terms of virtual magnitudes

$$p_2 + t_2 = \pi_2(\cdot) + t_2\pi_3(\cdot),$$

the quantity relationship between x_2 and the public good

$$t_2 x_2(\pi_2(\cdot), \pi_3(\cdot), \mu(\cdot)) = X_3(\pi_2(\cdot), \pi_3(\cdot), \mu(\cdot)) - \tilde{X}_3,$$

and the definition of virtual full income

$$\mu(\cdot) = w + \pi_3(\cdot)\tilde{X}_3.$$

Differentiating these equations with respect to t_2, using Cramer's rule and the Slutsky decomposition, we can solve for changes in each of the virtual magnitudes:

$$\begin{aligned}
\frac{\partial \pi_2}{\partial t_2} &= \frac{t_2 x_2 - (1-\pi_3)(C_{33} - t_2 C_{23}) - (1-\pi_3)(t_2 x_2)(t_2 x_{2\mu} - X_{3\mu})}{\Omega}, \\
\frac{\partial \pi_3}{\partial t_2} &= \frac{-x_2 + (1-\pi_3)(C_{32} - t_2 C_{22}) + (1-\pi_3)x_2(t_2 x_{2\mu} - X_{3\mu}))}{\Omega}, \\
\frac{\partial \mu}{\partial t_2} &= \frac{\partial \pi_3}{\partial t_2}\tilde{X}_3.
\end{aligned}$$

Substituting these three expressions into [A2], using the Slutsky decomposition again, and rearranging immediately yields equation [13].

A4. Verification that $\partial\hat{x}_2/\partial t_2 \lesseqgtr 0$ given assumption [15]

We have already established that

$$\frac{\partial\hat{x}_2}{\partial t_2} = \frac{x_2(t_2C_{22} - C_{23}) + (1-\pi_3)\Psi + (1-\pi_3)x_2\left[(C_{33} - t_2C_{32})x_{2\mu} + (t_2C_{22} - C_{23})X_{3\mu}\right]}{\Omega}.$$

Following the same steps for deriving a comparative static result using virtual magnitudes, it can be shown that

$$\frac{\partial\hat{X}_3}{\partial\tilde{X}_3} = \frac{(t_2C_{32} - C_{33}) + t_2\pi_3\left[(C_{23} - t_2C_{22})X_{3\mu} + (t_2C_{32} - C_{33})x_{2\mu}\right]}{\Omega},$$

and assumption [15] requires that $0 < \partial\hat{X}_3/\partial\tilde{X}_3 \leq 1$. To simplify notation, and without loss of generality, we normalize $t_2 = 1$ and let $A \equiv C_{32} - C_{33}$ and $B \equiv C_{23} - C_{22}$. Note that $A + B = \Omega > 0$, and recall that $\Psi < 0$. The two equations above simplify to

$$\frac{\partial\hat{x}_2}{\partial t_2} = \frac{-x_2B + (1-\pi_3)\Psi + (1-\pi_3)x_2\left[-Ax_{2\mu} - BX_{3\mu}\right]}{A+B} \qquad \text{[A3]}$$

and

$$\frac{\partial\hat{X}_3}{\partial\tilde{X}_3} = \frac{A + \pi_3\left[Ax_{2\mu} + BX_{3\mu}\right]}{A+B}. \qquad \text{[A4]}$$

Using the notation in [A4], satisfying the conditions in [15] requires both of the following:

$$A + \pi_3\left[Ax_{2\mu} + BX_{3\mu}\right] > 0 \qquad \text{[A5]}$$

and

$$\pi_3\left[Ax_{2\mu} + BX_{3\mu}\right] - B \leq 0. \qquad \text{[A6]}$$

With a bit of rearranging to [A3], it is straightforward to verify that

$$sign\left(\frac{\partial\hat{x}_2}{\partial t_2}\right) = sign\left(\pi_3\left[Ax_{2\mu} + BX_{3\mu}\right] - B + (1-\pi_3)\frac{\Psi}{x_2} - \left[Ax_{2\mu} + BX_{3\mu}\right]\right).$$

Because our objective is to show different possibilities, we can get some traction by assuming $\Psi \to 0$, as nothing rules out this possibility. In this case, we have

$$sign\left(\frac{\partial \hat{x}_2}{\partial t_2}\right) = sign\left(\underbrace{\pi_3\left[Ax_{2\mu} + BX_{3\mu}\right] - B}_{\leq 0} - \left[Ax_{2\mu} + BX_{3\mu}\right]\right), \qquad \text{[A7]}$$

where the inequality follows from [A6] This expression shows how the sign will depend primarily on the sign and magnitudes of price and income effects.

To focus on what is perhaps the most plausible scenario, let us assume that both x_2 and X_3 are normal (i.e., $x_{2\mu}, X_{3\mu} \geq 0$). In this case, it follows that if the two good are Hicksian substitutes (i.e., $A, B > 0$), the sign of [A7] is always negative. It only remains to show the possibility for [A7] to be positive without violating any of the conditions. Clearly, a necessary condition is $Ax_{2\mu} + BX_{3\mu} < 0$. Continuing to assume both goods are normal, this requires either A or B negative. But [A5] implies that we cannot satisfy the inequality and have $A < 0$. Hence $A > 0$ and $B < 0$ is a necessary condition, and this means that the two goods are Hicksian complements in the case where $0 > C_{22} > C_{32}(= C_{23}) > C_{33}$. Given that this is plausible, we therefore find nothing that rules out the possibility for [A7] to be positive.

A5. The relationship between the sign of $\varepsilon_{x_2^N,t_2}$ and $\varepsilon_{\hat{x}_2,t_2}$

We begin with the identity

$$x_2^N(t_2) = \hat{x}_2\big(t_2, (n-1)t_2 x_2^N(t_2)\big).$$

Differentiating yields

$$\frac{\partial x_2^N}{\partial t_2} = \frac{\partial \hat{x}_2}{\partial t_2} + \frac{\partial \hat{x}_2}{\partial \tilde{X}_3}(n-1)\left(x_2^N + t_2 \frac{\partial x_2^N}{\partial t_2}\right),$$

and rewriting with elasticities implies

$$\varepsilon_{x_2^N,t_2} = \frac{\varepsilon_{\hat{x}_2,t_2} + D}{1-D}, \qquad \text{[A8]}$$

where $D \equiv (n-1)\frac{\partial \hat{x}_3}{\partial \tilde{X}_3} \leq 0$, and the sign follows by the definition $\frac{\partial \hat{x}_3}{\partial \tilde{X}_3} = \frac{\partial \hat{X}_3}{\partial \tilde{X}_3} - 1$ combined with the assumption in [15]. Equation [A8] indicates that $\varepsilon_{x_2^N,t_2}$ shares the same sign as $\varepsilon_{\hat{x}_2,t_2}$ in all cases except for one, where $\varepsilon_{\hat{x}_2,t_2} > 0$ and the numerator of [A8] is negative, which is more likely to occur with large n, greater crowding out, or both. More generally, we find that nothing rules out the possibility for $\varepsilon_{x_2^N,t_2}$ to take either sign.

A6. Derivation of the relative magnitudes in Table 2

Let us assume $n \geq 2$. The conditions that define the Pareto optimal allocation in [8] can be written as follows:

$$\begin{aligned} \frac{U_3^*}{U_1^*} &= \frac{1}{p_1 n} \\ \frac{U_2^*}{U_1^*} &= \frac{p_2}{p_1}. \end{aligned}$$

The conditions that define the allocation consistent with the optimal dedicated tax in [A8] and [12] can be written in parallel fashion as

$$\frac{U_3^N}{U_1^N} = \frac{1}{p_1\left(n + (n-1)\varepsilon_{x_2^N,t_2}\right)}$$

$$\frac{U_2^N}{U_1^N} = \frac{p_2 + t_2\left(1 - \frac{1}{n + (n-1)\varepsilon_{x_2^N,t_2}}\right)}{p_1}$$

where the second comes from substituting the first into [12] and rearranging. Comparing these conditions, it follows that

$$\frac{U_3^N}{U_1^N} \gtreqless \frac{U_3^*}{U_1^*} \Leftrightarrow \varepsilon_{x_2^N,t_2} \lesseqgtr 0$$

and

$$\frac{U_2^N}{U_1^N} \gtreqless \frac{U_2^*}{U_1^*} \Leftrightarrow \varepsilon_{x_2^N,t_2} \gtreqless -1.$$

Three cases are then useful to consider:

$$\varepsilon_{x_2^N,t_2} \geq 0 \Rightarrow \frac{U_3^*}{U_1^*} \geq \frac{U_3^N}{U_1^N} \text{ and } \frac{U_2^*}{U_1^*} < \frac{U_2^N}{U_1^N}$$

$$\varepsilon_{x_2^N,t_2} \in (-1,0) \Rightarrow \frac{U_3^*}{U_1^*} < \frac{U_3^N}{U_1^N} \text{ and } \frac{U_2^*}{U_1^*} < \frac{U_2^N}{U_1^N}.$$

$$\varepsilon_{x_2^N,t_2} \leq -1 \Rightarrow \frac{U_3^*}{U_1^*} < \frac{U_3^N}{U_1^N} \text{ and } \frac{U_2^*}{U_1^*} \geq \frac{U_2^N}{U_1^N}.$$

The results in Table 2 follow by aligning each of these conditions with strict concavity with respect to each argument of the utility function, along with use of the budget constraint.